FREE NOTES · NCERT + JEE LEVEL

Three Dimensional Geometry

Direction cosines and ratios of a line, equation of a line in vector and Cartesian form, angle between two lines, and shortest distance between skew and parallel lines.

01 Direction cosines & direction ratios
02 Line through two points
03 Equation of a line
04 Angle between two lines
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01 Direction cosines & direction ratios

- If a line makes angles α, β, γ with the x, y, z axes: **l = cos α, m = cos β, n = cos γ**, and $l^2 + m^2 + n^2 = 1$.
- **Direction ratios** a, b, c are proportional to l, m, n: $l = a/\sqrt{(a^2 + b^2 + c^2)}$, etc.
- A line has two sets of DCs (l, m, n) and (−l, −m, −n).

02 Line through two points

DRs of PQ: $x_2 - x_1, y_2 - y_1, z_2 - z_1$

DCs: $(x_2 - x_1)/PQ, (y_2 - y_1)/PQ, (z_2 - z_1)/PQ$

03 Equation of a line

Given	Vector form	Cartesian form
Point a, parallel to b	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$	$(x - x_1)/a = (y - y_1)/b = (z - z_1)/c$
Two points a, b	$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$	$(x - x_1)/(x_2 - x_1) = (y - y_1)/(y_2 - y_1) = (z - z_1)/(z_2 - z_1)$

Any general point on the line: $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$. Make sure x, y, z have coefficient 1 before reading DRs.

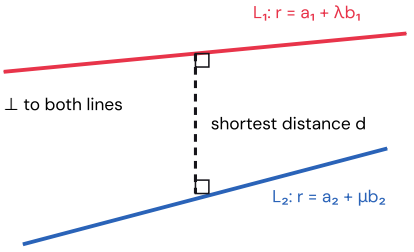
04 Angle between two lines

$$\cos\theta = |a_1a_2 + b_1b_2 + c_1c_2| \,/\, (\sqrt{(a_1^2+b_1^2+c_1^2)}\,\sqrt{(a_2^2+b_2^2+c_2^2)})$$

Vector form: $\cos\theta = |\mathbf{b_1}\cdot\mathbf{b_2}|/(|\mathbf{b_1}||\mathbf{b_2}|)$. Perpendicular: $a_1a_2 + b_1b_2 + c_1c_2 = 0$. Parallel: $a_1/a_2 = b_1/b_2 = c_1/c_2$.

05 Skew lines

Lines in space that are **neither parallel nor intersecting** (they lie in different planes). The shortest distance is along the common perpendicular.



Shortest distance between skew lines

06 Shortest distance

$$d = |(b_1 \times b_2)\cdot(a_2 - a_1)| \,/\, |b_1 \times b_2|$$

For skew lines $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$. $d = 0 \Rightarrow$ lines intersect.

$$d = |b \times (a_2 - a_1)| \,/\, |b|$$

For parallel lines $r = a_1 + \lambda b$ and $r = a_2 + \mu b$.

07 Formula sheet

DCs	$l^2 + m^2 + n^2 = 1$
Line	$r = a + \lambda b;\ (x-x_1)/a = (y-y_1)/b = (z-z_1)/c$
Angle	$\cos\theta = \lvert b_1\cdot b_2\rvert/(\lvert b_1\rvert\lvert b_2\rvert)$
SD skew	$\lvert(b_1\times b_2)\cdot(a_2-a_1)\rvert/\lvert b_1\times b_2\rvert$
SD parallel	$\lvert b\times(a_2-a_1)\rvert/\lvert b\rvert$

08 JEE Extra + Tricks

JEE LEVEL

- Shortest distance between skew lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$, $\vec{r} = \vec{a_2} + \mu\vec{b_2}$: $|(\vec{a_2} - \vec{a_1})\cdot(\vec{b_1} \times \vec{b_2})|/|\vec{b_1} \times \vec{b_2}|$; parallel lines: $|(\vec{a_2} - \vec{a_1}) \times \vec{b}|/|\vec{b}|$.
- Lines intersect iff shortest distance = 0 (coplanar and not parallel).
- JEE planes: $ax + by + cz + d = 0$; distance from point $|ax_1 + by_1 + cz_1 + d|/\sqrt{a^2 + b^2 + c^2}$; angle between planes via normals.
- Plane through intersection of two planes: $P_1 + \lambda P_2 = 0$; image of a point in a plane uses the same ratio trick as 2D.
- Line $\vec{r} = \vec{a} + \lambda\vec{b}$ lies in plane $\vec{r}\cdot\vec{n} = d$ iff $\vec{b}\cdot\vec{n} = 0$ and $\vec{a}\cdot\vec{n} = d$.

09 Solved examples

Q1. DCs of a line making equal angles with the axes.

$3l^2 = 1 \Rightarrow l = m = n = \pm 1/\sqrt{3}$.

Q2. Equation of the line through $(5, 2, -4)$ parallel to $3\hat{i} + 2\hat{j} - 8\hat{k}$.

$\vec{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$; $(x - 5)/3 = (y - 2)/2 = (z + 4)/(-8)$.

Q3. Angle between $(x + 3)/3 = (y - 1)/5 = (z + 3)/4$ and $(x + 1)/1 = (y - 4)/1 = (z - 5)/2$.

$\cos\theta = (3 + 5 + 8)/(\sqrt{50}\,\sqrt{6}) = 16/(10\sqrt{3}) = 8\sqrt{3}/15$.

Q4. SD between $\vec{r} = \hat{i} + 2\hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

$\vec{b_1} \times \vec{b_2} = -3\hat{i} + 0\hat{j} + 3\hat{k}$, $\vec{a_2} - \vec{a_1} = \hat{i} - 3\hat{j} - 2\hat{k} \Rightarrow |-3 - 6|/(3\sqrt{2}) = \mathbf{3/\sqrt{2}}$.

10 Common mistakes

- Reading DRs from (2x − 1)/3 as 3 (rewrite as (x − ½)/(3/2)).
- Using the skew-line formula for parallel lines ($b_1 \times b_2 = 0$).
- Forgetting the modulus in cos θ.
- Mixing up a (point) and b (direction) in r = a + λb.

11 Practice questions

1. If l = 1/2, m = 1/2, find n.
2. Are lines with DRs 1, 2, 3 and −2, −4, −6 parallel?
3. Find k if (x − 1)/(−3) = (y − 2)/(2k) = (z − 3)/2 and (x − 1)/(3k) = (y − 1)/1 = (z − 6)/(−5) are perpendicular.
4. Equation of the line through origin and (5, −2, 3).

ANSWERS

1) ±1/√2 · 2) Yes · 3) k = −10/7 · 4) x/5 = y/(−2) = z/3

12 Quick revision: 5 minutes before the exam

DIRECTION

- $l^2 + m^2 + n^2 = 1$.
- DRs ∝ DCs.

LINE

- r = a + λb.
- Coefficient of x, y, z = 1.

ANGLE

- ⊥: $\Sigma a_i a_j = 0$.
- ∥: ratios equal.

DISTANCE

- Skew: $|(b_1 \times b_2) \cdot (a_2 - a_1)| / |b_1 \times b_2|$.
- d = 0 ⇒ intersect.

Have a doubt? Ask the AI Doubt Solver at padhaiadda.netlify.app, or take the mock test on this chapter.