FREE NOTES · NCERT + JEE LEVEL

Application of Derivatives

Rate of change of quantities, increasing and decreasing functions, local maxima and minima using first and second derivative tests, absolute maxima and minima on closed intervals, and word problems on optimisation.

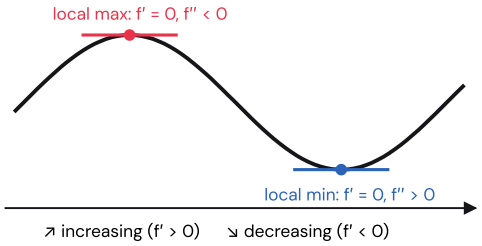
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01 Rate of change

dy/dx = rate of change of y w.r.t. x. If both depend on time: dy/dt = (dy/dx)(dx/dt).

Circle: $A = \pi r^2 \Rightarrow dA/dt = 2\pi r\, dr/dt$. Sphere: $V = (4/3)\pi r^3 \Rightarrow dV/dt = 4\pi r^2\, dr/dt$. Cube: $V = x^3 \Rightarrow dV/dt = 3x^2\, dx/dt$. Marginal cost = dC/dx, marginal revenue = dR/dx.

02 Increasing & decreasing functions



Sign of f′ decides rising or falling; extremes where f′ = 0

On an interval	f is
f′(x) > 0	Strictly increasing
f′(x) < 0	Strictly decreasing
f′(x) = 0	Constant

Method: find f′, solve f′ = 0, split the number line at these points, check the sign of f′ in each part.

03 Critical points

Points where f′(c) = 0 or f′ doesn't exist. Local maxima and minima can only occur at critical points (for a function defined on an open interval).

04 First derivative test

Sign of f′ as x crosses c	Result
+ to −	Local maximum
− to +	Local minimum
No change	Point of inflection (neither)

05 Second derivative test

$f'(c) = 0$ and $f''(c) < 0 \Rightarrow$ local maximum

$f'(c) = 0$ and $f''(c) > 0 \Rightarrow$ local minimum

$f''(c) = 0 \Rightarrow$ test fails, use the first derivative test

06 Absolute maximum & minimum on [a, b]

STEPS

1. Find all critical points in (a, b).
2. Evaluate f at these points and at the end points a, b.
3. The largest value is the absolute max; the smallest is the absolute min.

07 Optimisation (word problems)

1. Name the quantity to optimise (area, volume, cost…) and write it as a function of one variable using the given condition.
2. Differentiate, set equal to 0.
3. Use the second derivative to confirm max/min.
4. Answer what is asked (dimensions, value).

Standard results: rectangle of given perimeter has max area when it is a square; cylinder in a sphere, cone in a sphere (h = 4r/3 of sphere radius r), open box from a square sheet (cut x = side/6).

JEE extra: tangent at (x_1, y_1): $y - y_1 = f'(x_1)(x - x_1)$; normal: slope $-1/f'(x_1)$.

08 JEE Extra + Tricks

JEE LEVEL

- Tangent at (x_1, y_1): $y - y_1 = m(x - x_1)$; normal slope $-1/m$. Length of subtangent y/m, subnormal $y \cdot m$.
- Increasing if $f' \geq 0$ on interval; critical points where $f' = 0$ or undefined.
- Second derivative test: $f'' < 0 \Rightarrow$ local max, $f'' > 0 \Rightarrow$ local min; if $f'' = 0$ use first derivative test.
- Max/min word problems trick: for fixed perimeter, area of rectangle is max for a square; for fixed surface, cylinder volume max when $h = 2r$.
- Approximation: $\Delta y \approx f'(x)\Delta x$ (e.g. $\sqrt{26} \approx 5 + 1/10$).

09 Solved examples

Q1. Radius of a circle increases at 0.7 cm/s. Rate of change of circumference?

$C = 2\pi r \Rightarrow dC/dt = 2\pi \times 0.7 =$ **1.4π cm/s**.

Q2. Find intervals where $f(x) = 2x^3 - 3x^2 - 36x + 7$ is increasing/decreasing.

$f' = 6(x - 3)(x + 2)$. Increasing on $(-\infty, -2) \cup (3, \infty)$, decreasing on $(-2, 3)$.

Q3. Local max/min of $f(x) = x^3 - 6x^2 + 9x + 15$.

$f' = 3(x - 1)(x - 3)$; $f'' = 6x - 12$. $x = 1$: $f'' < 0 \Rightarrow$ max $f(1) = 19$. $x = 3$: $f'' > 0 \Rightarrow$ min $f(3) = 15$.

Q4. Two positive numbers with sum 15 and maximum product?

$P = x(15 - x)$, $P' = 15 - 2x = 0 \Rightarrow x = 7.5 \Rightarrow$ numbers **7.5 and 7.5**.

Q5. Absolute max/min of $f(x) = 2x^3 - 15x^2 + 36x + 1$ on $[1, 5]$.

$f' = 6(x - 2)(x - 3)$. $f(1) = 24$, $f(2) = 29$, $f(3) = 28$, $f(5) = 56 \Rightarrow$ max **56** at $x = 5$, min **24** at $x = 1$.

10 Common mistakes

- Forgetting the end points while finding absolute extrema.
- Calling every critical point a max or min.
- Not writing the function in one variable before differentiating.
- Wrong sign of f' in intervals (test a value in each).

11 Practice questions

1. Is $f(x) = e^x$ increasing on R?
2. Minimum value of $x^2 - 8x + 17$?
3. A sphere's volume grows at 10 cm^3/s. How fast is the radius growing when $r = 5$ cm?
4. Max area of a rectangle with perimeter 40.

ANSWERS

1) Yes ($f' = e^x > 0$) · 2) 1 at $x = 4$ · 3) $1/(10\pi)$ cm/s · 4) 100 (square of side 10)

12 Quick revision: 5 minutes before the exam

RATES

- dy/dt = (dy/dx)(dx/dt).
- Sphere dV/dt = $4\pi r^2$ dr/dt.

MONOTONIC

- $f' > 0$ increasing.
- $f' < 0$ decreasing.

EXTREMA

- $f' = 0$ then $f'' < 0$ max, > 0 min.
- Sign change + → − ⇒ max.

ABSOLUTE

- Check critical points + ends.

Have a doubt? Ask the AI Doubt Solver at padhaiadda.netlify.app, or take the mock test on this chapter.