FREE NOTES · NCERT + JEE LEVEL

Inverse Trigonometric Functions

Why we restrict domains, principal value branches of all six inverse functions, their graphs, principal values, and the key properties used to simplify expressions.

01 Why restrict the domain?
02 Principal value branches
03 Graphs
04 Finding principal values
05 Properties: self-cancelling
06 Properties: negative, reciprocal, sum
07 Simplifying tricks
08 JEE Extra + Tricks
09 Solved examples
10 Common mistakes
11 Practice questions
12 Quick revision

01 Why restrict the domain?

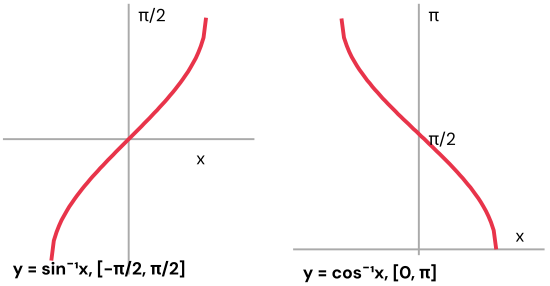
sin x is not one-one on R (sin 0 = sin π), so it has no inverse there. If we restrict sin to [−π/2, π/2], it becomes one-one and onto [−1, 1], and its inverse $\sin^{-1}$ exists. The chosen interval is the **principal value branch**.

$\sin^{-1}x \neq (\sin x)^{-1}$. $\sin^{-1}x$ means "the angle whose sine is x".

02 Principal value branches

Function	Domain	Range (principal branch)
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	R	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	R	$(0, \pi)$
$\sec^{-1}x$	$R - (-1, 1)$	$[0, \pi] - \{\pi/2\}$
$\operatorname{cosec}^{-1}x$	$R - (-1, 1)$	$[-\pi/2, \pi/2] - \{0\}$

03 Graphs



Principal branches of $\sin^{-1}x$ and $\cos^{-1}x$ (mirror of sin, cos in the line y = x)

04 Finding principal values

METHOD

Find the angle in the **principal range** with the given value. $\sin^{-1}(-½) = -\pi/6$ (not $7\pi/6$). $\cos^{-1}(-½) = \pi - \pi/3 = 2\pi/3$. $\tan^{-1}(-\sqrt{3}) = -\pi/3$.

05 Properties: self-cancelling

$\sin(\sin^{-1}x) = x,\ x \in [-1, 1]$ (always true)

$\sin^{-1}(\sin x) = x$ only if $x \in [-\pi/2, \pi/2]$

Example: $\sin^{-1}(\sin 3\pi/5) = \sin^{-1}(\sin 2\pi/5) = 2\pi/5$, because $3\pi/5$ is outside $[-\pi/2, \pi/2]$. Same idea for cos (range $[0, \pi]$) and tan.

06 Properties: negative, reciprocal, sum

Type	Results
Negative argument	$\sin^{-1}(-x) = -\sin^{-1}x$; $\tan^{-1}(-x) = -\tan^{-1}x$; $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$ $\cos^{-1}(-x) = \pi - \cos^{-1}x$; $\sec^{-1}(-x) = \pi - \sec^{-1}x$; $\cot^{-1}(-x) = \pi - \cot^{-1}x$
Reciprocal	$\sin^{-1}(1/x) = \operatorname{cosec}^{-1}x$; $\cos^{-1}(1/x) = \sec^{-1}x$; $\tan^{-1}(1/x) = \cot^{-1}x\ (x > 0)$
Complementary	$\sin^{-1}x + \cos^{-1}x = \pi/2$; $\tan^{-1}x + \cot^{-1}x = \pi/2$; $\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2$
Sum (JEE)	$\tan^{-1}x + \tan^{-1}y = \tan^{-1}[(x + y)/(1 - xy)],\ xy < 1$ $2\tan^{-1}x = \sin^{-1}(2x/(1+x^2)) = \cos^{-1}((1-x^2)/(1+x^2)) = \tan^{-1}(2x/(1-x^2))$

07 Simplifying tricks

- $\sqrt{(1 - x^2)}$ inside? Put $x = \sin\theta$ or $\cos\theta$.
- $\sqrt{(1 + x^2)}$? Put $x = \tan\theta$. $\sqrt{(x^2 - 1)}$? Put $x = \sec\theta$.
- $(1 - \cos x)/(1 + \cos x)$ type: use half-angle formulas.
- $(\cos x - \sin x)/(\cos x + \sin x) = \tan(\pi/4 - x)$.

08 JEE Extra + Tricks

JEE LEVEL

- $\sin^{-1}x + \cos^{-1}x = \pi/2$; $\tan^{-1}x + \cot^{-1}x = \pi/2$; $\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2$.
- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}((x + y)/(1 - xy))$ if $xy < 1$; add π if $xy > 1$ ($x, y > 0$).
- $2\tan^{-1}x = \sin^{-1}(2x/(1 + x^2)) = \cos^{-1}((1 - x^2)/(1 + x^2)) = \tan^{-1}(2x/(1 - x^2))$ (with ranges).
- $\sin^{-1}(\sin x) = x$ only for $x \in [-\pi/2, \pi/2]$; otherwise shift into principal range.
- $\cos^{-1}(-x) = \pi - \cos^{-1}x$; $\sin^{-1}(-x) = -\sin^{-1}x$; $\tan^{-1}(-x) = -\tan^{-1}x$.

09 Solved examples

Q1. Find $\tan^{-1}(1) + \cos^{-1}(-\tfrac{1}{2}) + \sin^{-1}(-\tfrac{1}{2})$.

$\pi/4 + 2\pi/3 - \pi/6 = \mathbf{3\pi/4}$.

Q2. Simplify $\tan^{-1}(\sqrt{(1+x^2)} - 1)/x$, $x \neq 0$.

$x = \tan\theta$: $(\sec\theta - 1)/\tan\theta = (1 - \cos\theta)/\sin\theta = \tan(\theta/2) \Rightarrow \mathbf{\tfrac{1}{2}\tan^{-1}x}$.

Q3. Find $\cos^{-1}(\cos 7\pi/6)$.

$7\pi/6 \notin [0, \pi]$; $\cos 7\pi/6 = \cos 5\pi/6 \Rightarrow \mathbf{5\pi/6}$.

Q4. Simplify $\tan^{-1}[(\cos x - \sin x)/(\cos x + \sin x)]$, $-\pi/4 < x < 3\pi/4$.

$= \tan^{-1}\tan(\pi/4 - x) = \mathbf{\pi/4 - x}$.

10 Common mistakes

- Giving an answer outside the principal range (e.g. $\sin^{-1}(-\tfrac{1}{2}) = 7\pi/6$).
- Writing $\sin^{-1}(\sin x) = x$ for every x.
- Treating $\sin^{-1}x$ as $1/\sin x$.
- Using $\cos^{-1}(-x) = -\cos^{-1}x$ (it's $\pi - \cos^{-1}x$).

11 Practice questions

1. Principal value of $\operatorname{cosec}^{-1}(2)$.
2. Domain of $\sin^{-1}(2x)$.
3. Find $\sin(\pi/3 - \sin^{-1}(-\tfrac{1}{2}))$.
4. If $\sin^{-1}x = \pi/5$, find $\cos^{-1}x$.

ANSWERS

1) $\pi/6$ · 2) $[-\tfrac{1}{2}, \tfrac{1}{2}]$ · 3) 1 · 4) $3\pi/10$

12 Quick revision: 5 minutes before the exam

RANGES

- $\sin^{-1}$, $\tan^{-1}$: around 0.
- $\cos^{-1}$, $\cot^{-1}$: 0 to π.

NEGATIVE

- sin, tan, cosec: odd.
- cos, sec, cot: π − ....

SUM

- $\sin^{-1} + \cos^{-1} = \pi/2$.
- $\tan^{-1} + \cot^{-1} = \pi/2$.

CAREFUL

- $\sin^{-1}(\sin x) = x$ only in range.
- Substitutions: sin θ, tan θ, sec θ.

Have a doubt? Ask the AI Doubt Solver at padhaiadda.netlify.app, or take the mock test on this chapter.