FREE NOTES · NCERT + JEE/NEET LEVEL

# Motion in a Plane

From vectors to projectiles, circular motion and the classic river-boat and rain-umbrella relative velocity problems. Diagrams, shortcuts and solved numericals with every topic.

01 Scalar vs vector
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## 01 Scalar vs vector

| | Scalar | Vector |
|---|---|---|
| Tells you | Only magnitude (how much) | Magnitude + direction |
| Adding | Add numbers directly | Triangle / parallelogram law |
| Examples | Mass, time, distance, speed, work, energy, temperature | Displacement, velocity, acceleration, force, momentum |

**SPECIAL VECTORS**

- **Equal vectors:** same magnitude, same direction (can be placed anywhere).
- **Negative vector:** same magnitude, opposite direction: −A.
- **Null (zero) vector:** magnitude 0, any direction. A + (−A) = 0.
- **Unit vector:** magnitude 1, shows only direction: $\hat{A} = A/|A|$.

**NOTE**

Electric current has a direction but is a **scalar**, because it does not add by vector rules (triangle law). To be a vector, a quantity must have direction and also obey vector addition.

## 02 Adding vectors

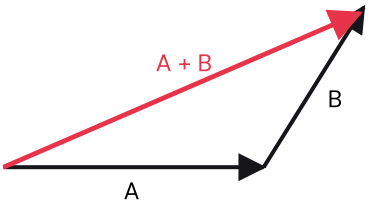
**Triangle law:** place the tail of B at the head of A. The arrow from the tail of A to the head of B is A + B.

**Parallelogram law:** draw both from one point. The diagonal of the parallelogram is the resultant.

**MAGNITUDE & DIRECTION**

$$R = \sqrt{(A^2 + B^2 + 2AB\cos\theta)}$$

$$\tan\alpha = B\sin\theta / (A + B\cos\theta)$$




Triangle law: head to tail

| Angle θ | Resultant R |
|---|---|
| 0° (same direction) | A + B (maximum) |
| 90° | √(A² + B²) |
| 180° (opposite) | \|A − B\| (minimum) |
| A = B at angle θ | 2A cos(θ/2); at θ = 120°, R = A |

**RANGE RULE**

$|A - B| \leq |A + B| \leq A + B$. Three vectors can add to zero only if the third lies within the range of the first two (they can form a triangle).

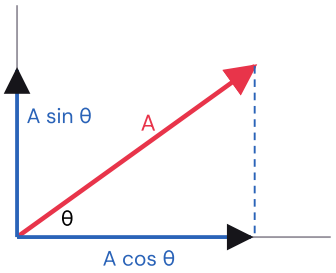
**Subtraction:** $A - B = A + (-B)$, magnitude $\sqrt{(A^2 + B^2 - 2AB\cos\theta)}$. Vector addition is commutative (A + B = B + A) and associative; subtraction is not commutative.

## 03 Resolution & unit vectors

Splitting a vector into two perpendicular parts is **resolution**. If it makes angle θ with the x-axis:

$$A_x = A\cos\theta,\ A_y = A\sin\theta$$

$$A = \sqrt{(A_x^2 + A_y^2)},\ \tan\theta = A_y/A_x$$




$A_x$ and $A_y$ together rebuild A

**Unit vectors** $\hat{i}, \hat{j}, \hat{k}$ point along the x, y, z axes. Any vector: $A = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$, magnitude $\sqrt{(A_x^2 + A_y^2 + A_z^2)}$.

**DIRECTION COSINES (3D)**

Angles with the axes α, β, γ: $\cos\alpha = A_x/A$, $\cos\beta = A_y/A$, $\cos\gamma = A_z/A$, and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

## 04 Dot & cross product

**DOT (SCALAR) PRODUCT**

$$A \cdot B = AB\cos\theta$$

$$= A_xB_x + A_yB_y + A_zB_z$$

$\hat{i}\cdot\hat{i} = 1$, $\hat{i}\cdot\hat{j} = 0$. Perpendicular ⇒ dot = 0. Example: work W = F · s.

**CROSS (VECTOR) PRODUCT**

$$|A \times B| = AB\sin\theta$$

Direction: right-hand rule (perpendicular to both). $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$; $\hat{i} \times \hat{i} = 0$. A × B = −B × A. Parallel ⇒ cross = 0. Example: torque τ = r × F.

To find the angle: $\cos\theta = (A \cdot B)/(AB)$. Component of A along B: $A\cos\theta = (A \cdot B)/B$.

## 05 Velocity & acceleration in 2D

Position vector $r = x\,\hat{i} + y\,\hat{j}$. Displacement $\Delta r = r_2 - r_1$.

$$v = dr/dt = v_x\,\hat{i} + v_y\,\hat{j},\ a = dv/dt = a_x\,\hat{i} + a_y\,\hat{j}$$

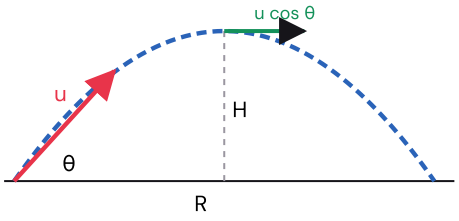
Velocity always points along the **tangent** to the path. For constant a, apply per axis: $v = u + at$, $r = r_0 + ut + \frac{1}{2}at^2$.

**THE KEY IDEA**

Motions along x and y are **independent of each other**. Every projectile problem is solved using this idea.

## 06 Projectile motion

An object thrown into the air with only gravity acting on it. No horizontal force ⇒ horizontal velocity is constant. Vertically a = -g.




At the top only the horizontal velocity u cos θ remains

**GROUND TO GROUND**

Time of flight: $T = 2u \sin\theta / g$

Max height: $H = u^2 \sin^2\theta / (2g)$

Range: $R = u^2 \sin 2\theta / g$

Path: $y = x \tan\theta - g x^2 / (2u^2 \cos^2\theta)$

The path is a parabola.

| Results to remember | |
|---|---|
| Maximum range | at $\theta = 45°$, $R_{max} = u^2/g$, then $H = R/4$ |
| Equal range | $\theta$ and $(90° - \theta)$ give the same range (e.g. 30° and 60°) |
| H and R | $\tan\theta = 4H/R$ |
| H and T | $H = gT^2/8$ |
| At the top | $v = u\cos\theta$, $a = g$ (down), velocity $\perp$ acceleration |
| Kinetic energy at the top | $K\cos^2\theta$ ($K$ = initial KE) |

## 07 Horizontal projection (thrown horizontally from height h)

Time: $T = \sqrt{(2h/g)}$ (independent of u!)

Range: $R = u\sqrt{(2h/g)}$

Speed at ground: $v = \sqrt{(u^2 + 2gh)}$, $\tan\beta = \sqrt{(2gh)}/u$ (below horizontal)

**CLASSIC QUESTION**

Fire one bullet horizontally and drop another from the same height at the same moment: **both hit the ground together**, because their vertical motion is identical.

**INCLINED PLANE (JEE)**

On an incline of angle $\alpha$, thrown at $\theta$ to the incline: $T = 2u\sin\theta/(g\cos\alpha)$, $R = 2u^2\sin\theta\cos(\theta + \alpha)/(g\cos^2\alpha)$ (up the incline). Maximum range when $\theta = 45° - \alpha/2$.

## 08 Uniform circular motion

Moving in a circle at constant speed. The speed stays the same but the direction changes every instant, so there **is an acceleration**, directed toward the centre.

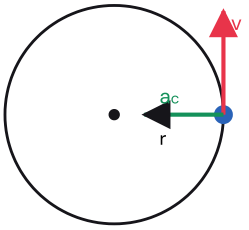
**FORMULAS**

Angular velocity: $\omega = d\theta/dt = 2\pi/T = 2\pi f$

$v = \omega r$

$$a_c = v^2/r = \omega^2 r$$

Centripetal acceleration always points to the centre. T = time for one round, f = 1/T.




v along the tangent, $a_c$ toward the centre: $v \perp a$

**NOTE**

In uniform circular motion the speed is constant, but **both velocity and acceleration keep changing** (because of direction). It is accelerated motion.

## 09 Non-uniform circular motion

If the speed also changes, there are two accelerations:

Tangential: $a_t = dv/dt = r\,\alpha$ (changes speed)

Centripetal: $a_c = v^2/r$ (changes direction)

$$a = \sqrt{(a_t^2 + a_c^2)}$$

α = angular acceleration (rad/s²). Angular equations look like the linear ones: $\omega = \omega_o + \alpha t$, $\theta = \omega_o t + \frac{1}{2}\alpha t^2$, $\omega^2 = \omega_o^2 + 2\alpha\theta$.

## 10 Relative velocity in 2D

$v_{AB} = v_A - v_B$ (vector subtraction).

**RAIN AND UMBRELLA**

Rain falls vertically at $v_r$, a man walks at $v_m$. To him the rain appears to come at $v_{rm} = v_r - v_m$.

Tilt the umbrella forward at θ from the vertical: $\tan\theta = v_m/v_r$.

**RIVER-BOAT**

River width d, river speed $v_r$, boat speed (relative to water) $v_b$.

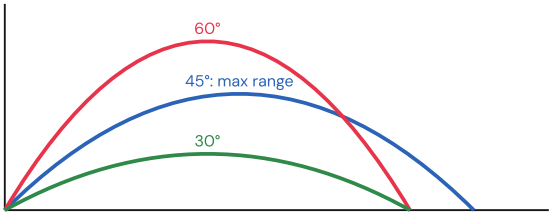
Minimum time: head straight across ⇒ $t = d/v_b$, drift $= v_r d/v_b$

Reach the point directly opposite (zero drift): head upstream with $\sin\theta = v_r/v_b$, $t = d/\sqrt{(v_b^2 - v_r^2)}$

## 11 JEE Extra + Tricks

**JEE LEVEL**

- Projectile: $T = 2u\sin\theta/g$, $H = u^2\sin^2\theta/2g$, $R = u^2\sin2\theta/g$; R max = $u^2/g$ at 45°; $H_{max} = R_{max}/2$.
- Complementary angles ($\theta$, 90° − $\theta$) give the same range; product of their times $T_1T_2 = 2R/g$.
- Trajectory: $y = x\tan\theta - gx^2/(2u^2\cos^2\theta) = x\tan\theta\,(1 - x/R)$.
- River crossing: shortest time ⇒ swim perpendicular, $t = d/v_s$; shortest path (zero drift) ⇒ $\sin\alpha = v_r/v_s$ upstream.
- Uniform circular motion: $a = v^2/r = \omega^2 r$ towards centre; non-uniform adds tangential $a_t = dv/dt$; net $a = \sqrt{(a_c^2 + a_t^2)}$.




30° and 60° give the same range (complementary angles)

Projectiles at 30°, 45°, 60° with same speed

## 12 Solved numericals ($g = 10\ m/s^2$)

**Q1. $A = 3\hat{i} + 4\hat{j}$ and $B = \hat{i} - 2\hat{j}$. Find $|A + B|$ and $A \cdot B$.**
$A + B = 4\hat{i} + 2\hat{j} \Rightarrow |A + B| = \sqrt{20} = 2\sqrt{5}$. $A \cdot B = 3 - 8 = -5$. **Ans: $2\sqrt{5}$, $-5$**

**Q2. Two equal forces F have a resultant also equal to F. Angle between them?**
$F = 2F\cos(\theta/2) \Rightarrow \cos(\theta/2) = ½ \Rightarrow \theta/2 = 60°$. **Ans: 120°**

**Q3. A ball is thrown at 20 m/s at 30°. Find T, H and R.**
$T = 2 \times 20 \times ½/10 = 2$ s. $H = 400 \times ¼/20 = 5$ m. $R = 400 \times \sin 60°/10 = 40 \times 0.866 \approx 34.6$ m. **Ans: 2 s, 5 m, ≈ 34.6 m**

**Q4. A stone is thrown horizontally at 15 m/s from an 80 m cliff. How far does it land and at what speed?**
$T = \sqrt{(160/10)} = 4$ s. $R = 15 \times 4 = 60$ m. $v_y = 40$ m/s $\Rightarrow v = \sqrt{(225 + 1600)} = \sqrt{1825} \approx 42.7$ m/s. **Ans: 60 m, ≈ 42.7 m/s**

**Q5. A projectile's range is 4 times its maximum height. Angle of projection?**
$\tan\theta = 4H/R = 4H/4H = 1 \Rightarrow$ **Ans: 45°**

**Q6. A stone on a 1 m string is whirled at 2 revolutions per second. Centripetal acceleration?**
$\omega = 2\pi \times 2 = 4\pi$ rad/s. $a = \omega^2 r = 16\pi^2 \approx 158$ m/s². **Ans: ≈ 158 m/s²**

**Q7. Rain falls vertically at 30 m/s and a man walks east at 10 m/s. How should he hold his umbrella?**
$\tan\theta = 10/30 = 1/3 \Rightarrow \theta \approx 18.4°$ from the vertical, tilted toward the east (forward). **Ans: tilt ≈ 18° forward**

**Q8. A 400 m wide river flows at 3 m/s; a boat's speed is 5 m/s. (a) Minimum crossing time? (b) Time to reach the point directly opposite?**
(a) $t = 400/5 = 80$ s, drift $= 3 \times 80 = 240$ m. (b) $t = 400/\sqrt{(25 - 9)} = 400/4 = 100$ s. **Ans: 80 s, 100 s**

## 13 Common mistakes

- Taking velocity at the top of a projectile as 0. Only the vertical part is 0; $u\cos\theta$ remains.
- Using the range formula ($u^2 \sin 2\theta/g$) when thrown from a height. It is only for ground to ground.
- Thinking acceleration is 0 in uniform circular motion.
- Adding vector magnitudes directly: 3 N + 4 N is not always 7 N.
- Tilting the umbrella backward. While walking, rain appears to come from the front, so tilt it **forward**.
- Mixing up the minimum-time and zero-drift river cases.

## 14 Formula sheet

| | |
|---|---|
| Resultant | $R = \sqrt{(A^2 + B^2 + 2AB\cos\theta)}$ |
| Components | $A_x = A\cos\theta$, $A_y = A\sin\theta$ |
| Dot / cross | $A\cdot B = AB\cos\theta$, $\lvert A\times B\rvert = AB\sin\theta$ |
| Projectile T, H, R | $2u\sin\theta/g$, $u^2\sin^2\theta/2g$, $u^2\sin 2\theta/g$ |
| Max range | $u^2/g$ at $45°$ |
| Trajectory | $y = x\tan\theta - gx^2/(2u^2\cos^2\theta)$ |
| Horizontal projection | $T = \sqrt{(2h/g)}$, $R = u\sqrt{(2h/g)}$ |
| Circular motion | $v = \omega r$, $\omega = 2\pi/T$, $a_c = v^2/r = \omega^2 r$ |
| Non-uniform | $a = \sqrt{(a_t^2 + a_c^2)}$ |
| Rain-umbrella | $\tan\theta = v_m/v_r$ |
| River: min time / zero drift | $d/v_b$ / $d/\sqrt{(v_b^2 - v_r^2)}$ |

## 15 Practice questions

1. Are $A = 2\hat{i} + 3\hat{j} - \hat{k}$ and $B = \hat{i} - \hat{j} + \hat{k}$ perpendicular? Check.
2. A projectile's maximum height is 20 m. Time of flight?
3. At what angle are the range and maximum height equal?
4. A car at 36 km/h takes a turn of radius 50 m. Centripetal acceleration?
5. A stone is thrown horizontally at 10 m/s from 5 m height. Range?

**ANSWERS**

1) $A\cdot B = 2 - 3 - 1 = -2 \neq 0$, not perpendicular · 2) $T = \sqrt{(8H/g)} = 4$ s · 3) $\tan\theta = 4 \Rightarrow \theta \approx 76°$ · 4) $100/50 = 2$ m/s² · 5) $10 \times 1 = 10$ m

## 16 Quick revision: 5 minutes before the exam

**VECTORS**

- $R = \sqrt{(A^2 + B^2 + 2AB\cos\theta)}$.
- $|A - B| \le R \le A + B$.
- $A \cdot B = 0 \Rightarrow$ perpendicular; $A \times B = 0 \Rightarrow$ parallel.
- Equal vectors at $120° \Rightarrow R = A$.

**PROJECTILE**

- Horizontal v constant, vertical $a = -g$.
- $T = 2u\sin\theta/g$, $H = u^2\sin^2\theta/2g$, $R = u^2\sin 2\theta/g$.
- Max range at 45°; $\theta$ and $90° - \theta$ give the same range.
- Thrown horizontally: $T = \sqrt{(2h/g)}$.

**CIRCULAR**

- $v = \omega r$, $a_c = v^2/r = \omega^2 r$.
- Uniform: speed constant, velocity changes.
- Non-uniform: $a = \sqrt{(a_t^2 + a_c^2)}$.

**RELATIVE**

- $v_{AB} = v_A - v_B$.
- Umbrella: $\tan\theta = v_m/v_r$, tilt forward.
- River: min time $d/v_b$; zero drift $\sin\theta = v_r/v_b$.

Have a doubt? Ask the AI Doubt Solver at padhaiadda.netlify.app, or take the mock test on this chapter.