FREE NOTES · NCERT + JEE LEVEL

Limits & Derivatives

Intuitive idea of a limit, left and right limits, algebra of limits, standard limits of polynomial, rational and trigonometric functions, derivative from first principles, rules of differentiation and derivatives of standard functions.

01 The idea of a limit
02 Algebra of limits
03 Standard limits
04 Trigonometric limits
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01 The idea of a limit

$\lim_{x \to a} f(x) = L$ means f(x) gets as close to L as we like when x gets close to a (but $x \neq a$). f(a) may not even exist.

- **Left-hand limit** $\lim_{x \to a^-} f(x)$ and **right-hand limit** $\lim_{x \to a^+} f(x)$.
- The limit exists only if LHL = RHL.
- Example: f(x) = |x|/x has LHL = −1, RHL = 1 at 0, so no limit.

02 Algebra of limits

$$\lim [f \pm g] = \lim f \pm \lim g;\ \lim [f \cdot g] = \lim f \cdot \lim g$$

$$\lim [f/g] = \lim f \,/\, \lim g \ (\text{if } \lim g \neq 0);\ \lim [k\, f] = k \lim f$$

Polynomials: just substitute. Rational: substitute; if you get 0/0, factorise and cancel, or rationalise.

03 Standard limits

$$\lim_{x \to a} (x^n - a^n)/(x - a) = n\, a^{n-1}$$

$\lim_{x \to 0} (e^x - 1)/x = 1;\ \lim_{x \to 0} \log(1 + x)/x = 1$ (JEE)

Example: $\lim_{x \to 1} (x^{15} - 1)/(x^{10} - 1)$ = (15/10) = 3/2 (divide both by x - 1).

04 Trigonometric limits

$$\lim_{x \to 0} \sin x / x = 1$$

$\lim_{x \to 0} (1 - \cos x)/x = 0;\ \lim_{x \to 0} \tan x / x = 1$

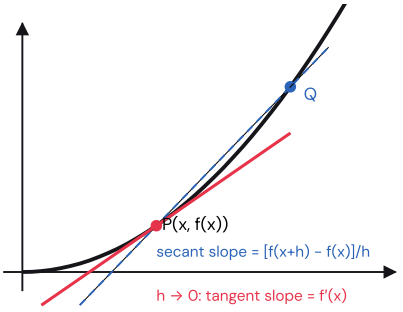
$\lim_{x \to 0} \sin ax / bx = a/b$

x must be in radians. Proved using the sandwich theorem (cos x < sin x / x < 1).

05 Derivative: what it means

The derivative f′(x) is the **instantaneous rate of change** of f, i.e. the slope of the tangent to the curve at that point.

Physics link: velocity = derivative of displacement; acceleration = derivative of velocity.



The secant becomes the tangent as h → 0

06 Derivative from first principles

$$f'(x) = \lim_{h \to 0} [f(x+h) - f(x)] / h$$

Example $f(x) = x^2$: $[(x+h)^2 - x^2]/h = 2x + h \to \mathbf{2x}$. Example $f(x) = \sin x$: $[\sin(x+h) - \sin x]/h = 2\cos(x + h/2)\sin(h/2)/h \to \mathbf{\cos x}$.

07 Rules & standard derivatives

Rule	Formula
Sum	$(u \pm v)' = u' \pm v'$
Product (Leibniz)	$(uv)' = u'v + uv'$
Quotient	$(u/v)' = (u'v - uv')/v^2$
Constant multiple	$(ku)' = k\,u'$

f(x)	f′(x)	f(x)	f′(x)
x^n	$n\,x^{n-1}$	c	0
sin x	cos x	cos x	−sin x
tan x	$\sec^2 x$	cot x	$-\mathrm{cosec}^2 x$
sec x	sec x tan x	cosec x	−cosec x cot x

08 JEE Extra + Tricks

JEE LEVEL

- Standard limits: $\lim \frac{\sin x}{x} = 1$, $\lim \frac{1 - \cos x}{x^2} = \frac{1}{2}$, $\lim \frac{e^x - 1}{x} = 1$, $\lim \frac{a^x - 1}{x} = \ln a$, $\lim \frac{\ln(1 + x)}{x} = 1$, $\lim \frac{x^n - a^n}{x - a} = na^{n-1}$.
- 1^∞ form: $\lim f^g = e^{\lim g(f - 1)}$.
- L'Hôpital's rule (0/0 or ∞/∞): differentiate numerator and denominator separately.
- Expansions near 0: $\sin x \approx x - x^3/6$, $\cos x \approx 1 - x^2/2$, $\tan x \approx x + x^3/3$, $e^x \approx 1 + x + x^2/2$, $\ln(1 + x) \approx x - x^2/2$.
- $\lim_{x \to \infty}$ of rational function: compare highest powers.

09 Solved examples

Q1. $\lim_{x\to 2} (x^2 - 4)/(x - 2)$.

$(x - 2)(x + 2)/(x - 2) \to$ **4**.

Q2. $\lim_{x\to 0} \sin 4x / \sin 2x$.

$(\sin 4x/4x)\cdot 4x / [(\sin 2x/2x)\cdot 2x] \to$ **2**.

Q3. $\lim_{x\to 0} (\sqrt{(1 + x)} - 1)/x$.

Rationalise: $x/[x(\sqrt{(1+x)} + 1)] \to$ **½**.

Q4. Differentiate $(x + 1)(x^2 - 3)$.

$(x^2 - 3) + (x + 1)(2x) =$ **$3x^2 + 2x - 3$**.

Q5. Differentiate $(x + \cos x)/\tan x$.

$[(1 - \sin x) \tan x - (x + \cos x) \sec^2 x]/\tan^2 x$.

10 Common mistakes

- Writing the limit equals f(a) always (only when f is continuous).
- Using degrees in sin x/x.
- Product rule as (uv)′ = u′v′ (wrong).
- Quotient rule with the numerator order reversed (u′v − uv′, not uv′ − u′v).
- Derivative of cos x as sin x (missing minus).

11 Formula sheet & practice

Standard	$(x^n - a^n)/(x - a) \to na^{n-1}$
Trig	$\sin x/x \to 1$
First principle	$\lim [f(x+h) - f(x)]/h$
Product / quotient	$u'v + uv'; (u'v - uv')/v^2$

1. $\lim_{x\to 0} (1 - \cos 2x)/x^2$.
2. Derivative of x^{-3}.
3. Derivative of $\sin x \cos x$.

4. Find $f'(1)$ if $f(x) = x^{100} + x^{99} + \ldots + x + 1$.

ANSWERS

1) 2 · 2) $-3x^{-4}$ · 3) cos 2x · 4) 5050

12 Quick revision: 5 minutes before the exam

LIMITS

- LHL = RHL ⇒ limit exists.
- 0/0 ⇒ factorise/rationalise.

STANDARD

- $(x^n - a^n)/(x - a) \to na^{n-1}$.
- $\sin x/x \to 1$ (radians).

DERIVATIVE

- $f' = \lim [f(x+h) - f(x)]/h$.
- Slope of tangent.

RULES

- $(uv)' = u'v + uv'$.
- $(u/v)' = (u'v - uv')/v^2$.

Have a doubt? Ask the AI Doubt Solver at padhaiadda.netlify.app, or take the mock test on this chapter.